Dynamical scaling in branching models for seismicity

Eugenio Lippiello,[1] Cataldo Godano[2] and Lucilla de Arcangelis[3]
[1] *Department of Physical Sciences, University of Naples "Federico II", 80125 Napoli, Italy*
[2] *Department of Environmental Sciences and CNISM, Second University of Naples, 81100 Caserta, Italy*
[3] *Department of Information Engineering and CNISM, Second University of Naples, 81031 Aversa (CE), Italy*

We propose a branching process based on a dynamical scaling hypothesis relating time and mass. In the context of earthquake occurrence, we show that experimental power laws in size and time distribution naturally originate solely from this scaling hypothesis. We present a numerical protocol able to generate a synthetic catalog with an arbitrary large number of events. The numerical data reproduce the hierarchical organization in time and magnitude of experimental inter-event time distribution.

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Stochastic branching models have been used since a long time in the description of a large variety of social and physical phenomena ranging from biological evolution or genealogy [1], to nuclear or chemical reactions [2]. In the last years branching processes have been widely studied in seismicity and actually they provide one of the most efficient tools for earthquake forecasting [3]. Within this approach, one treats seismicity as a marked point process in time $\{M_i(t_i)\}$ where t_i is the occurrence time of an event with mass (magnitude) M_i and one assumes that each event can trigger future ones according to a two point conditional rate $\rho(M(t)|M_i(t_i))$. Given a history of past events $\{M_i(t_i)\}$, then, the rate of events of magnitude M at time t is given by

$$\rho(M(t)|\{M_i(t_i)\}) = \sum_{i:t_i<t} \rho(M(t)|M_i(t_i)) + \mu P(M) \quad (1)$$

where μ is a constant rate of independent sources and $P(M)$ their magnitude distribution. In the epidemic type aftershock sequences (ETAS) model [4], widely used in seismology, for any couple of events i and j, magnitude of event i is independent of previous events

$$\rho(M_i(t_i)|M_j(t_j)) = P(M_i) g(t_i - t_j; M_j) \quad (2)$$

and for the magnitude distribution $P(M_i)$ and the propagator $g(t_i - t_j; M_j)$ one uses three well known experimental observations:

i) The magnitude distribution follows an exponential law $P(M) \sim 10^{-bM}$ usually referred as the Gutenberg-Richter (GR) law [5], where $b \simeq 1$;

ii) The Omori law [6] states that the number of aftershocks $n(t)$ decays in time as $n(t) \sim (t+c)^{-p}$ with $p \simeq 1$;

iii) The aftershock number is exponentially related to the mainshock magnitude. This last law combined with the Omori law gives $g(t_i - t_j, M_j) \propto 10^{\alpha M_j}(t_i - t_j + c)^{-p}$, where $\alpha \sim b$ [7].

The main statistical properties of the ETAS model have been reviewed in a series of papers (see for instance ref. [8]). Ultraviolet and infrared cut-offs have to be necessarily introduced to make the theory convergent. Whereas a large magnitude cut-off can be expected on physical grounds, more questionable is the existence of a minimum magnitude M_{inf} [9]. This problem has been removed by a self-similar version of the ETAS model recently introduced by Vere-Jones [10]. In this model the decoupling (2) between magnitude and time is still assumed but a multiplicative factor is considered $\rho(M_i(t_i)|M_j(t_j)) = P(M_i)g(t_i - t_j; M_j)S(M_i - M_j)$ where $S(M_i - M_j) = 10^{-d|M_i - M_j|}$ introduces magnitude correlations: for $d > 0$ large daughter earthquakes tend to occur after large mother earthquakes. Saichev and Sornette have suggested a physical explanation for this magnitude correlation based on faults branching [11].

In this paper we show that the above mentioned magnitude correlations do not need to be introduced via an ad hoc term, as $S(M_i - M_j)$, but naturally originate from a more general scaling relation. More precisely we assume that the magnitude difference $M_i - M_j$ fixes a characteristic time scale

$$\tau_{ij} = k 10^{b(M_j - M_i)} \quad (3)$$

so that the conditional rate is magnitude independent when time is rescaled by τ_{ij} and k is a constant measured in seconds

$$\rho(M_i(t_i)|M_j(t_j)) = F\left[\frac{t_i - t_j}{\tau_{ij}}\right]. \quad (4)$$

With the only constraint that $F(x)$ can be normalized, we recover the statistical features of earthquake occurrence: Omori law, GR law and scaling behaviour of the interevent time distribution [12, 13]. Furthermore, using Eq.s (1,4) in a numerical code we are able to generate, in few hours of CPU time, a synthetic catalog with the same number of events as 30 years California catalog. Experimental and numerical catalogs are found to exhibit the same time and magnitude organization.

In order to verify the existence of magnitude correlations we consider the ANSS California catalog [14]

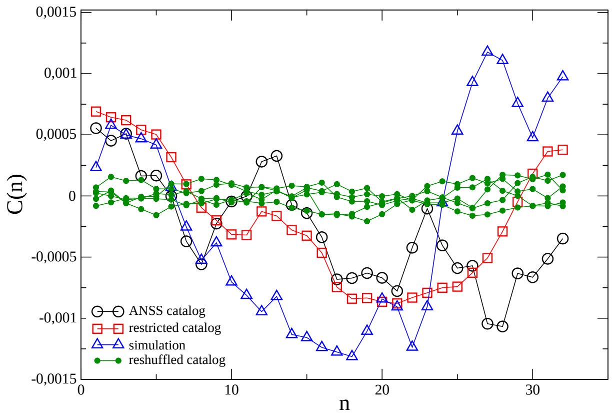

FIG. 1: (Color online) Function $C(n)$ versus n for the ANSS catalog ($\circ$), the restricted catalog ($\square$) and the simulated catalog ($\triangle$). $C(n)$ evaluated for five different realizations of reshuffled ANSS catalog (dots).

containing $N = 9586$ $M > 3$ earthquakes. We divide the catalog in $N_L = N/L$ subsets each containing $L = 125$ events and define the quantity $\delta M_j = (1/L)\sum_{i=j*L+1}^{j*L+L} M_i - (1/N)\sum_{i=1}^{N} M_i$, representing the deviation of the average magnitude in the j-th subset with respect to the average over the entire catalog. We then calculate the quantity $C(n)$ defined as

$$C(n) = \frac{1}{l}\sum_{j=n+1}^{n+l}\frac{1}{N_L - n_{max}}\sum_{i=1}^{N_L-n_{max}}\delta M_i \delta M_{i+j} \quad (5)$$

where $n_{max} = 32$ is the maximum "distance" between subsets considered. The advantage of correlating average quantities δM_j is to reduce fluctuations and the sum over j further smoothens statistical noise. In absence of magnitude correlations, δM_i and δM_{i+j} are statistically independent quantities and $C(n)$ does not depend on n and fluctuates around $C(n) = 0$. This situation can be realized by evaluating $C(n)$ after reshuffling the magnitudes. Using 100000 realizations of the reshuffled catalog, we find that the distribution of $C(n)$ exhibits gaussian behaviour centered in zero with standard deviation $\sigma = 0.0001$. Figure 1, conversely, shows that $C(n)$ computed for the real catalog has a regular trend with amplitude several times larger than σ, clearly indicating the existence of magnitude correlations. Therefore the behaviour of $C(n)$ for the real catalog is a signature of a well defined earthquake magnitude organization.

In order to check that the results of Fig. 1 are not a spurious effect of short term aftershock incompleteness [15], we use the method proposed by Helmstetter et al [7] stating that, after a main shock of magnitude M_M at time t_M, the completeness level $M > 3$ is recovered only after a time $t_C = t_M + 10^{(M_M-7.5)/0.75}$. We then construct a restricted catalog by neglecting all events occurring within a time interval $[t_M, t_C]$ after each event with $M > 1$, obtaining a catalog complete for $M > 3$ and containing $N = 8502$ events. The evaluation of $C(n)$ (Fig.1) now shows that correlations do not disappear but $C(n)$ exhibits again a regular trend with values significantly different than zero. The comparison of $C(n)$ in the original and restricted catalog indicates that magnitude correlations are not therefore an artifact of catalog incompleteness but should be attributed to a physical effect due to earthquake interactions. These interactions are introduced in our approach by means of the scaling assumption (4). In this way we are able to reproduce, at least at a qualitative level, the experimental behaviour of $C(n)$ as can be seen in Fig.1.

Before discussing the details of our numerical procedure, we explore the consequences of our scaling assumption (4). We first notice that the total number of daughter earthquakes, conditioned to the occurrence of a mother earthquake of magnitude M_0 at time t_0, is given by $\int_{t_0}^{\infty} dt\rho(M(t)|M_0(t_0)) = k10^{-b(M-M_0)}\int_0^{\infty} dx F(x)$. Since $F(x)$ is normalizable, we recover the GR behaviour independently of the specific form of $F(x)$. On the basis of this observation, a mother earthquake at time t_0 will be distributed according to GR law and therefore the occurrence rate of magnitude M triggered events at time t is then given by $\rho(M, t - t_0) = \int_{M_{inf}}^{M_{sup}} \rho(M(t)|M_0(t_0))P(M_0)dM_0$ which leads to

$$\rho(M, t - t_0) \propto \frac{10^{-bM}}{(t - t_0)}\int_{10^{-b(M_{sup}-M)(t-t_0)}}^{10^{b(M-M_{inf})(t-t_0)}} F(z)dz \quad (6)$$

We first observe that ultraviolet and infrared cut-offs are not necessary anymore. Indeed, in Eq.(6) one can arbitrary set $M_{inf} \to -\infty$ and $M_{sup} \to \infty$ obtaining $\rho(M, t - t_0) \propto \frac{10^{-bM}}{(t-t_0)}$. Then one recovers, as a direct consequence of the only assumption (4), beside the GR also the Omori law that conversely are assumed a priori in the ETAS model. The above result, that hold for quite arbitrary $F(x)$, suggests that these two fundamental laws, generally considered as independent laws in seismicity, can be strictly related to a more general scaling behaviour. Furthermore Eq.(6) shows that a change in M_{sup} or M_{inf} only corresponds to a time rescaling and therefore the only effect is on the parameter k in (3).

We now construct with our approach a synthetic catalog to be compared with the experimental one. In a numerical protocol one assumes at initial time $t_0 = 0$ a single event of arbitrary magnitude chosen in a fixed range $[M_{inf}, M_{sup}]$. Time is then increased by a unit step $t = t_0 + 1$, a trial magnitude is randomly chosen in the interval $[M_{inf}, M_{sup}]$ and Eqs.(1,4) give the probability to have an earthquake in the time window $(t_0, t_0 + 1)$. If this probability is larger than a random number between 0 and 1, an earthquake takes place, its magnitude and occurrence time are stored and used for the evaluation of probability for future events. Time is then increased and in this way one constructs a synthetic catalog

of N_e events. The term μ in Eq.(1) represents an additional source of earthquakes Poissonian distributed in time with a magnitude chosen from the GR distribution with $b = 0.8$. Other choices for $P(M)$ do not sensibly affect the statistical properties of the simulated catalog.

Following this protocol, we generate sequences of 15000 events using a power law form for $F(z) = A/(z^\lambda + \gamma)$ with $M_{inf} = 1$ and $M_{sup} = 8$. We compute magnitude distribution $P(M)$ and intertime distribution $D(\Delta t, M_L)$ where Δt is the time distance between successive events with magnitude greater than a given threshold M_L. Extended analysis of experimental catalogs have shown [12, 13] that the intertime distribution is a fundamental quantity to characterize the magnitude and time organization of earthquakes. In fact, indicating with $P_C(M)$ the cumulative magnitude distribution inside the considered region, one observes

$$D(\Delta t, M_L) = P_C(M_L) f(P_C(M_L)\Delta t) \qquad (7)$$

where f is a universal function, independent on M_L and on the geographical region indicating a well defined hierarchical organization of earthquake occurrence [16]. The numerical distributions are compared with the experimental data from the ANSS Catalog. For different values of λ, it is always possible to find a set of parameters A, γ, b, μ such that numerical data reproduce, on average, earthquake statistical features both in time and in magnitude. The parameter k is fixed a posteriori in order to obtain the collapse between numerical and experimental time. We have also performed simulations for different values of M_{inf} and M_{sup} obtaining similar results and confirming that changes in the magnitude range only produce time rescaling.

In Fig.2 we plot the experimental and numerical $D(\Delta t, M_L)$ considering two different values of λ ($\lambda = 1.2$ and 5) and M_L ($M_L = 1.5$ and 2.5). Data for different values of the parameters follow a universal curve and the same collapse is obtained for other values of $\lambda > 1$. The accordance between experimental and numerical curves (inset of Fig.2) indicates that the hypothesis of dynamical scaling is able to reproduce two fundamental properties of seismic occurrence, namely the GR law and Eq. (7), independently of the details of $F(z)$.

The ETAS model corresponds to a particular choice for $F(z)$, i.e. $\gamma = 0$ and $\lambda = p \simeq 1$. We want to stress the important difference due to the presence of a non-zero γ. First, the constant γ removes the problematic need of an infrared (ultraviolet) cut-off in the ETAS model, whereas in our approach M_{inf} and M_{sup} are irrelevant variables. Second, the constant γ gives rise to the observed magnitude correlation. Indeed, for a given mainshock of magnitude M_j at time t_j, at each time ($t_i > t_j$) it is possible to define a sufficiently large magnitude difference ΔM such that, if $M_j - M_i > \Delta M$, we have that z^λ is negligible with respect to γ and therefore $F[(t_i - t_j)/\tau_{ij}] \simeq A/\gamma$.

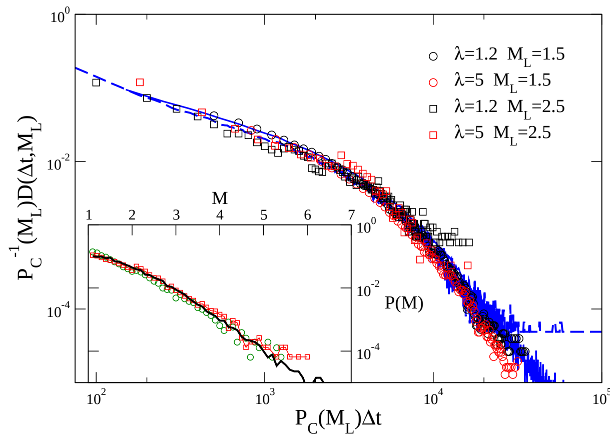

FIG. 2: (Color online) The intertime distribution with $F(z) = A/(z^\lambda + \gamma)$, with two values of $\lambda = 1.2, 5$ and $M_L = 1.5, 2.5$. Continuous and broken curve are the experimental $D(\Delta t, M_L)$ with $M_L = 1.5$ and $M_L = 2.5$. For $\lambda = 1.2$ ($\lambda = 5$) we set $k = 210sec$ ($k = 420sec$), $A = 1.4\ 10^{-4}sec^{-1}$ ($A = 1.9\ 10^{-4}sec^{-1}$), $\mu = 4\ 10^{-7}$ ($\mu = 1.510^{-6}$), $\gamma = 1$ ($\gamma = 0.1$) and $b = 1$. In the inset the magnitude distribution of the experimental (black line) and numerical catalog with $\lambda = 1.2$ (red $\circ$) and $\lambda = 5$ (green $\Box$).

In other words after a large event, the probability of big quakes is raised.

We have also performed more extensive simulations using for $F(z)$ an exponential behaviour

$$F(z) = \frac{A}{e^z - 1 + \gamma} \qquad (8)$$

Eq.(8) states that two events of magnitude M_i and M_j are correlated over a characteristic time τ_{ij} and become independent when $t_i - t_j > \tau_{ij}$. As a consequence, only a small fraction of previous events can affect the probability of future earthquakes so that, after a certain time, Earth crust loses memory of previous seismicity. This aspect is perhaps more realistic with respect to the idea, contained in a power law correlation, that events are all correlated with each other and also gives rise to important implications for seismic forecasting. The construction of seismic catalogs, indeed, dates back to about 50 years, and according to Eq.(8) one can have good estimates of seismic hazard without considering previous seismicity. This is no longer true if one assumes a power law time decorrelation. We want also to point out that a general state-rate formulation [17] gives rise to correlations between earthquakes that decay exponentially in time. We finally observe, that taking into account only a fraction of previous events in the evaluation of conditional probabilities, the numerical procedure considerably speeds up. In the case of long range temporal correlations, CPU time grows with the number of events as N_e^2, whereas in the case of an exponential tail the growth is linear in N_e. For this reason, assuming the functional form (8) one can simulate very large sequences of events.

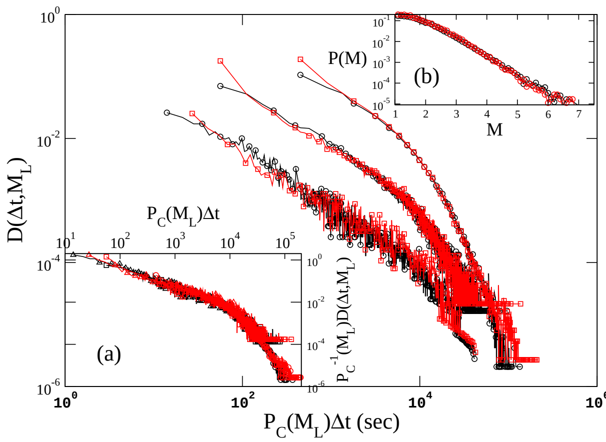

FIG. 3: (Color online) The intertime distribution as function of $\Delta t P_c(M_L)$ obtained using Eq.(8) (black circle ○) and compared with the experimental distributions (red □) for three different values of M_L ($M_L = 1.5, 2.5, 3.5$, from top to bottom). We set $k = 4.9\ 10^4 sec$, $A = 6.1\ 10^{-5} sec^{-1}$, $\mu = 2\ 10^{-5}$, $\gamma = 0.1$. In inset (a) collapse of the three curves for different M_L following the scaling of Eq.(7) and in inset (b) the experimental (red) and numerical (black) magnitude distribution.

In particular for a different choice of parameters, one can construct synthetic catalogs containing the same number of events ($N_e = 245000$ with $M \geq 1.5$) of the experimental California Catalog. In Fig. 3 we compare numerical and experimental distributions $D(\Delta t, M_L)$ for three different values of M_L. For each value of M_L, the numerical curve reproduces the experimental data and fulfills Eq.(7) (inset (a) in Fig.(3)). We have also evaluated the behaviour of $C(n)$ using the numerical catalog finding qualitative agreement with experimental results (Fig.1). Finally, the numerical magnitude distribution $P(M)$ fits very well the experimental one (inset (b) in Fig.(3)).

After fixing k, we express numerical time unit in seconds and we observe that the numerical catalog corresponds to a period of about $9.9\ 10^9 sec \simeq 30$ years. Our model is therefore able to construct a synthetic catalog covering about 30 years, containing about the same number of events and displaying the same statistical organization in magnitude and time of occurrence as real California Catalog. The high efficiency of the model in reproducing past seismicity indicates that the model is a good tool for earthquake forecasting. In fact, given a seismic history, Eq.(1) together with Eq.s(4, 8) gives the rate of occurrence of magnitude M earthquakes at time t inside a considered geographic region. We want to point out that similar extended analysis has never been performed in seismicity. It is, indeed, the first time that a stochastic model is able to produce a synthetic catalog with all the features of the real experimental intertime and magnitude distributions. Recently, Saichev and Sornette [21] have calculated the intertime distributions with an analytical approach to the ETAS models. Under the assumption that an earthquake can have at most one first generation aftershock they find for the intertime distribution a behaviour different from Eq.(7).

We finally observe that also spatial organization of seismic events reveals some kind of scale invariance [18, 19, 20]. This indicates that also spatial distribution originates from a critical behaviour of the Earth crust suggesting that a dynamical scaling hypothesis as in Eq.(4) can also work if one appropriately introduces spatial dependencies. In this way it would be possible to construct seismic hazard maps.

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